

# COMPLEX SYMMETRY OF COMPOSITION OPERATORS INDUCED BY INVOLUTIVE BALL AUTOMORPHISMS

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**ABSTRACT.** Suppose  $\mathcal{H}$  is a weighted Hardy space of analytic functions on the unit ball  $\mathbb{B}_n \subset \mathbb{C}^n$  such that the composition operator  $C_\psi$  defined by  $C_\psi f = f \circ \psi$  is bounded on  $\mathcal{H}$  whenever  $\psi$  is a linear fractional self-map of  $\mathbb{B}_n$ . If  $\varphi$  is an involutive Moebius automorphism of  $\mathbb{B}_n$ , we find a conjugation operator  $\mathcal{J}$  on  $\mathcal{H}$  such that  $C_\varphi = \mathcal{J}C_\psi^*\mathcal{J}$ . The case  $n = 1$  answers a question of Garcia and Hammond.

## 1. INTRODUCTION

Let  $\mathbb{B}_n$  denote the open unit ball of  $\mathbb{C}^n$ . A linear fractional self-map  $\psi$  of  $\mathbb{B}_n$  is a map of the form

$$(1.1) \quad \psi(z) = \frac{Az + B}{\langle z, C \rangle + d}$$

where  $A$  is a linear operator on  $\mathbb{C}^n$ , with vectors  $B, C \in \mathbb{B}_n$  and  $d$  a complex number. Fix a vector  $a \in \mathbb{B}_n$ . Let  $P_a$  be the orthogonal projection of  $\mathbb{C}^n$  onto the complex line generated by  $a$  and let  $Q_a = I - P_a$ . Setting  $s_a = (1 - |a|^2)^{1/2}$ , we denote by  $\varphi_a : \mathbb{B}_n \rightarrow \mathbb{B}_n$  the linear fractional map

$$(1.2) \quad \varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}$$

which by Section 2.2.1 of Rudin [9] is the involutive Moebius automorphism of  $\mathbb{B}_n$  that interchanges 0 and  $a$ .

Suppose  $\mathcal{H}$  is a weighted Hardy space on  $\mathbb{B}_n$  as introduced in [2]; that is a Hilbert space of analytic functions on  $\mathbb{B}_n$  such that the monomials  $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$  for  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  form an orthogonal basis for  $\mathcal{H}$ . Additionally, the monomials  $z^\alpha$  satisfy  $\frac{\|z^\alpha\|}{\|z^\alpha\|_{H^2}} = \frac{\|z^\gamma\|}{\|z^\gamma\|_{H^2}}$  whenever  $|\alpha| = |\gamma|$  for  $\gamma \in \mathbb{N}^n$ , where  $|\gamma| = \gamma_1 + \dots + \gamma_n$  and  $\|\cdot\|$  and  $\|\cdot\|_{H^2}$  are the norms in  $\mathcal{H}$  and the classical Hardy space  $H^2(\mathbb{B}_n)$  respectively. The sequence  $\beta_m = \frac{\|z^\alpha\|}{\|z^\alpha\|_{H^2}}$  for  $m = |\alpha|$  determines  $\mathcal{H}$ : the inner product on  $\mathcal{H}$  is given by

$$(1.3) \quad \langle f, g \rangle = \sum_{m=0}^{\infty} \langle f_m, g_m \rangle_{H^2} \beta_m^2$$

where  $f_m, g_m$  are the homogeneous polynomials of degree  $m$  in the homogeneous expansions  $f = \sum f_m$  and  $g = \sum g_m$  on  $\mathbb{B}_n$  respectively. If  $\psi$  is an analytic self-map of  $\mathbb{B}_n$ , then the composition operator  $C_\psi$  is defined by  $C_\psi f = f \circ \psi$  for  $f \in \mathcal{H}$ . When  $\mathcal{H}$  is the classical Hardy space or a weighted Bergman space on  $\mathcal{D} = \mathbb{B}_1$ , then

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the boundedness of  $C_\psi$  on  $\mathcal{H}$  follows from the Littlewood Subordination Theorem. But for  $n \geq 2$ , there exist unbounded composition operators even on the classical spaces [2]. For our purpose, it is sufficient to assume that  $\mathcal{H}$  is a weighted Hardy space such that  $C_\psi$  is bounded on  $\mathcal{H}$  whenever  $\psi$  is a linear fractional self-map of  $\mathbb{B}_n$ . From here onwards  $\mathcal{H}$  shall always denote such a space. Many of the classical Hilbert spaces of analytic functions on  $\mathbb{B}_n$  satisfy this requirement: For any real  $s > 0$ , let  $H_s$  be the Hilbert space of analytic functions on  $\mathbb{B}_n$  with reproducing kernel function

$$K_w^s(z) = \frac{1}{(1 - \langle w, z \rangle)^s}$$

for  $w, z \in \mathbb{B}_n$ . These spaces are called generalized weighted Bergman spaces by Zhao and Zhu [10]. Recently Le [8] showed that  $C_\psi$  is bounded on  $H_s$  for  $s > 0$  whenever  $\psi$  is a linear fractional self-map of  $\mathbb{B}_n$ . The space  $H_n$  is the classical Hardy space  $H^2(\mathbb{B}_n)$  and  $H_s$  is the weighted Bergman space  $A_{s-n-1}^2(\mathbb{B}_n)$  for  $s > n$ . When  $n \geq 2$ , then  $H_1$  is the Drury-Arveson space from multi-variable operator theory [1].

A bounded operator  $T$  on  $\mathcal{H}$  is called *complex symmetric* if  $T$  has a self-transpose matrix representation with respect to some orthonormal basis of  $\mathcal{H}$ . This is equivalent to the existence of a *conjugation* (i.e., a conjugate-linear, isometric involution)  $C$  such that  $T = CT^*C$ ; and  $T$  is called *C-symmetric*. If  $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$  then termwise complex conjugation  $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$  is a conjugation operator on  $\mathbb{C}^n$ . The general study of complex symmetric operators on Hilbert spaces was initiated by Garcia, Putinar and Wogen [4][5][6][7].

The complex symmetry of weighted composition operators ( $M_h C_\psi$ ; where  $M_h$  is multiplication by an analytic function  $h$  on  $\mathbb{B}_n$ ) was recently studied by Garcia and Hammond [3] for  $n = 1$ . A complete description of complex symmetric weighted composition operators remains an open problem (even in the case  $h \equiv 1$ ). Normal operators are the most obvious examples of complex symmetric operators and the characterization of normal composition operators on  $H_s$  for  $s > 0$  follows from Theorem 8.2 [2]:

**Proposition 1.1.** *Let  $\psi$  be an analytic self-map of  $\mathbb{B}_n$ . Then  $C_\psi$  is normal on  $H_s$  for  $s > 0$  if and only if  $\psi(z) = Az$  for some normal linear operator  $A$  on  $\mathbb{C}^n$  with  $\|A\| \leq 1$ .*

For a linear operator  $V$  on  $\mathbb{C}^n$  with  $\|V\| \leq 1$ , we denote by  $C_V$  the composition operator  $C_{\phi_V}$  where  $\phi_V(z) = Vz$  for  $z \in \mathbb{B}_n$ . Since  $\phi_V$  is a linear fractional map of the form (1.1), the operator  $C_V$  is bounded on  $\mathcal{H}$  and  $C_V^* = C_{V^*}$ . Define the conjugate-linear operator  $J : \mathcal{H} \rightarrow \mathcal{H}$  by  $(Jf)(z) = \overline{f(\bar{z})}$  for  $f \in \mathcal{H}$  and  $z \in \mathbb{B}_n$ . That  $J$  is a conjugation on  $\mathcal{H}$  follows from (1.3) and the fact that  $J$  is isometric on  $H^2(\mathbb{B}_n)$ . The following result is useful for constructing examples of complex symmetric composition operators.

**Proposition 1.2.** *If  $V$  is a complex symmetric linear operator on  $\mathbb{C}^n$  with  $\|V\| \leq 1$ , then  $C_V$  is complex symmetric on  $\mathcal{H}$ .*

*Proof.* First suppose that  $V$  has a symmetric matrix representation with respect to the standard orthonormal basis of  $\mathbb{C}^n$ . Then  $V$  is complex symmetric with respect to the usual complex conjugation  $\bar{z}$  for  $z \in \mathbb{C}^n$ ; that is  $\overline{V^*z} = Vz$ . Hence  $C_V$  is  $J$ -symmetric because for any  $f \in \mathcal{H}$  and  $z \in \mathbb{C}^n$ , we have

$$(JC_V^*Jf)(z) = f(\overline{V^*z}) = f(Vz) = (C_V f)(z).$$

In general, let  $U_V$  be a unitary operator on  $\mathbb{C}^n$  such that  $W = U_V^* V U_V$  has a symmetric matrix with respect to the standard basis of  $\mathbb{C}^n$ . Then  $C_V$  is unitarily equivalent to  $C_W$ ; that is  $C_V = C_{U_V^*} C_W C_{U_V}$  where  $C_{U_V}$  is unitary on  $\mathcal{H}$ . Hence if we define the conjugation  $\mathcal{J}_V$  on  $\mathcal{H}$  by  $\mathcal{J}_V = C_{U_V^*} J C_{U_V}$ , it follows that  $C_V$  is  $\mathcal{J}_V$ -symmetric.  $\square$

It is known that all  $2 \times 2$  complex matrices are complex symmetric [4]. For  $n = 2$  it follows that  $C_V$  is complex symmetric on  $\mathcal{H}$  for any linear operator  $V$  on  $\mathbb{C}^2$ . As a consequence of Proposition 1.1 and Proposition 1.2, if  $V$  is complex symmetric on  $\mathbb{C}^n$  but not normal, then  $C_V$  has the same properties on  $\mathcal{H}$ .

## 2. THE COMPLEX SYMMETRY OF $C_{\varphi_a}$

Not every complex symmetric composition operator is of the form  $C_V$  for some linear operator  $V$  on  $\mathbb{C}^n$ . The composition operator  $C_{\varphi_a}$  is bounded on  $\mathcal{H}$  and complex symmetric for all  $a \in \mathbb{B}_n$ . This is because  $C_{\varphi_a} \circ C_{\varphi_a} = I$ , and Theorem 2 [7] states that operators algebraic of degree 2 are complex symmetric. Our main goal is to construct a conjugation  $\mathcal{J}_a$  such that  $C_{\varphi_a}$  is  $\mathcal{J}_a$ -symmetric on  $\mathcal{H}$ . The particular case  $n = 1$  resolves a problem of Garcia and Hammond [3]: *If  $\varphi$  is an involutive disk automorphism, find an explicit conjugation  $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$  such that  $C_\varphi = \mathcal{J} C_\varphi^* \mathcal{J}$ .*

For any  $a \in \mathbb{B}_n$ , denote by  $W_a : \mathcal{H} \rightarrow \mathcal{H}$  the unitary part in the polar decomposition of  $C_{\varphi_a}$  on  $\mathcal{H}$ . We need two lemmas for constructing the conjugation  $\mathcal{J}_a$  from  $W_a$ . The first one is a general result about the unitary part in the polar decomposition of a linear involution on a Hilbert space.

**Lemma 2.1.** *If  $T$  is an invertible operator on a Hilbert space with  $T^2 = I$  and with polar decomposition  $T = U|T|$ , then  $U^2 = I$ . In particular,  $U$  is self-adjoint.*

*Proof.* We first prove that  $|T|^{-1} = |T^*|$ . It is known that if two positive operators  $A$  and  $B$  commute then  $\sqrt{A}$  and  $\sqrt{B}$  also commute. Hence

$$(T^* T)(T T^*) = (T T^*)(T^* T)$$

implies that  $|T||T^*| = |T^*||T|$ . So  $(|T^*||T|)^2 = |T^*|^2 |T|^2 = (T T^*)(T^* T) = I$ . Since the product of two commuting positive operators is again positive, the uniqueness of the square root of a positive operator implies  $|T||T^*| = |T^*||T| = I$ . We also note that the observation  $T^*(T^* T) = (T T^*)T^*$  implies  $T^*|T| = |T^*|T^*$ . Now since  $U = T^*|T|$  it follows that

$$U^2 = (|T^*|T^*)(T^*|T|) = |T^*||T| = I.$$

Since  $U$  is unitary,  $U^2 = I$  implies  $U^* = U$ .  $\square$

By Lemma 2.1, the unitary operator  $W_a$  is self-adjoint and satisfies  $W_a^2 = I$  on  $\mathcal{H}$  for any  $a \in \mathbb{B}_n$ .

**Lemma 2.2.** *If  $a \in \mathbb{B}_n \cap \mathbb{R}^n$  then  $JW_a = W_a J$ .*

*Proof.* The polar decomposition  $C_{\varphi_a} = W_a |C_{\varphi_a}|$  implies that  $W_a = C_{\varphi_a}^* |C_{\varphi_a}|$ . We first show that  $J C_{\varphi_a} = C_{\varphi_a} J$  for  $a \in \mathbb{B}_n \cap \mathbb{R}^n$ . For  $a = (a_1, \dots, a_n) \in \mathbb{B}_n \cap \mathbb{R}^n$  and  $z = (z_1, \dots, z_n) \in \mathbb{B}_n$ , we have  $J\langle z, a \rangle = \langle \bar{z}, a \rangle = \langle z, a \rangle$ . If  $j = 1, \dots, n$ , by (1.2)

and  $P_az = \frac{\langle z, a \rangle}{\langle a, a \rangle}a$  and  $Q_az = z - \frac{\langle z, a \rangle}{\langle a, a \rangle}a$ , the  $j$ -th component function of  $\varphi_a$  denoted by  $[\varphi_a]_j$  is

$$\begin{aligned} [\varphi_a]_j(z) &= \left[ \frac{a - P_az - s_a Q_az}{1 - \langle z, a \rangle} \right]_j = \left[ \frac{(1 - \frac{\langle z, a \rangle}{\langle a, a \rangle} + s_a \frac{\langle z, a \rangle}{\langle a, a \rangle})a - s_az}{1 - \langle z, a \rangle} \right]_j \\ &= \frac{(1 - \frac{\langle z, a \rangle}{\langle a, a \rangle} + s_a \frac{\langle z, a \rangle}{\langle a, a \rangle})a_j - s_az_j}{1 - \langle z, a \rangle}. \end{aligned}$$

By  $J\langle z, a \rangle = \langle z, a \rangle$  it follows that  $J[\varphi_a]_j = [\varphi_a]_j$  and hence for any  $j = 1, \dots, n$  we have

$$JC_{\varphi_a}Jz_j = JC_{\varphi_a}z_j = J[\varphi_a]_j(z) = [\varphi_a]_j(z) = C_{\varphi_a}z_j.$$

Since  $C_{\varphi_a}$  and  $J$  are both multiplicative operators on  $\mathcal{H}$ , we have  $JC_{\varphi_a}Jz^\alpha = C_{\varphi_a}z^\alpha$  for any multi-index  $\alpha \in \mathbb{N}^n$ . Therefore  $JC_{\varphi_a}J = C_{\varphi_a}$  for  $a \in \mathbb{B}_n \cap \mathbb{R}^n$ . From this it follows that  $JC_{\varphi_a}^* = C_{\varphi_a}^*J$  and hence  $J|C_{\varphi_a}| = |C_{\varphi_a}|J$ . Since  $W_a = C_{\varphi_a}^*|C_{\varphi_a}|$ , we get  $JW_a = W_aJ$  for  $a \in \mathbb{B}_n \cap \mathbb{R}^n$ .  $\square$

So for  $a \in \mathbb{B}_n \cap \mathbb{R}^n$ , lemmas 2.1 and 2.2 imply that the conjugate-linear isometry defined by  $\mathcal{J}_a = JW_a$  is an involution on  $\mathcal{H}$ ; that is  $\mathcal{J}_a^2 = (W_aJ)(JW_a) = W_a^2 = I$ . So  $\mathcal{J}_a$  is a conjugation on  $\mathcal{H}$  and furthermore:

**Theorem 2.3.** *If  $a \in \mathbb{B}_n \cap \mathbb{R}^n$  then  $C_{\varphi_a}$  is  $\mathcal{J}_a$ -symmetric on  $\mathcal{H}$ .*

*Proof.* From Lemma 2.2 and the self-adjointness of  $W_a$  we get

$$\begin{aligned} C_{\varphi_a}^*\mathcal{J}_a &= C_{\varphi_a}^*JW_a = C_{\varphi_a}^*W_aJ = C_{\varphi_a}^*W_a^*J = C_{\varphi_a}^*|C_{\varphi_a}|C_{\varphi_a}J \\ &= W_aC_{\varphi_a}J = W_aJC_{\varphi_a} = \mathcal{J}_aC_{\varphi_a} \end{aligned}$$

and hence that  $C_{\varphi_a}$  is  $\mathcal{J}_a$ -symmetric when  $a \in \mathbb{B}_n \cap \mathbb{R}^n$ .  $\square$

For the general case  $a \in \mathbb{B}_n$ , we will show that  $C_{\varphi_a}$  is unitarily equivalent to  $C_{\varphi_{\tilde{a}}}$  for some  $\tilde{a} \in \mathbb{B}_n \cap \mathbb{R}^n$ . For  $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ , denote by  $U_\Theta : \mathcal{H} \rightarrow \mathcal{H}$  the bounded composition operator defined by  $(U_\Theta f)(z) = f(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$  for  $f \in \mathcal{H}$ . Then  $U_\Theta$  is unitary on  $H^2(\mathbb{B}_n)$  and hence also on  $\mathcal{H}$  by (1.3). If the vector  $-\Theta = (-\theta_1, \dots, -\theta_n) \in \mathbb{R}^n$ , then  $U_\Theta^* = U_{-\Theta}$ . Now fix  $a \in \mathbb{B}_n$  and choose  $\Theta \in \mathbb{R}^n$  such that  $\tilde{a} = (e^{i\theta_1}a_1, \dots, e^{i\theta_n}a_n) \in \mathbb{R}^n$ . Since  $\langle U_\Theta^*z, a \rangle = \sum_{j=1}^n e^{-i\theta_j}z_j\bar{a}_j = \langle z, \tilde{a} \rangle$  and  $s_a = s_{\tilde{a}}$ , for  $j = 1, \dots, n$  we have

$$\begin{aligned} U_\Theta^*C_{\varphi_a}U_\Theta z_j &= U_\Theta^*C_{\varphi_a}e^{i\theta_j}z_j = U_\Theta^*e^{i\theta_j}[\varphi_a]_j = U_\Theta^*e^{i\theta_j} \frac{(1 - \frac{\langle z, a \rangle}{\langle a, a \rangle} + s_a \frac{\langle z, a \rangle}{\langle a, a \rangle})a_j - s_az_j}{1 - \langle z, a \rangle} \\ &= \frac{(1 - \frac{\langle U_\Theta^*z, a \rangle}{\langle a, a \rangle} + s_a \frac{\langle U_\Theta^*z, a \rangle}{\langle a, a \rangle})e^{i\theta_j}a_j - s_a e^{i\theta_j}U_\Theta^*z_j}{1 - \langle U_\Theta^*z, a \rangle} \\ &= \frac{(1 - \frac{\langle z, \tilde{a} \rangle}{\langle \tilde{a}, \tilde{a} \rangle} + s_{\tilde{a}} \frac{\langle z, \tilde{a} \rangle}{\langle \tilde{a}, \tilde{a} \rangle})\tilde{a}_j - s_{\tilde{a}}z_j}{1 - \langle z, \tilde{a} \rangle} = C_{\varphi_{\tilde{a}}}z_j \end{aligned}$$

which implies that  $U_\Theta^*C_{\varphi_a}U_\Theta = C_{\varphi_{\tilde{a}}}$  on  $\mathcal{H}$ . To define the conjugation  $\mathcal{J}_a$  when  $a \in \mathbb{B}_n$ , we let  $\mathcal{J}_a = U_\Theta\mathcal{J}_{\tilde{a}}U_\Theta^*$ . Hence we have proved

**Theorem 2.4.** *If  $a \in \mathbb{B}_n$  then  $C_{\varphi_a}$  is  $\mathcal{J}_a$ -symmetric on  $\mathcal{H}$ , where  $\mathcal{J}_a = U_\Theta\mathcal{J}_{\tilde{a}}U_\Theta^*$  and  $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$  is chosen such that  $\tilde{a} = (e^{i\theta_1}a_1, \dots, e^{i\theta_n}a_n) \in \mathbb{R}^n$ .*

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